Chapter 1

Rectangular Cartesian Coordinates

It is sometimes said that Descartes' work *La Geometric* marks the turning point between medieval and modern mathematics. He demonstrated the interplay between algebra and geometry, tying together these two branches of mathematics. Due to poor health as a child he was always permitted to remain in bed as long as he wished. He maintained this habit throughout his life and did his most productive thinking while tying in bed in the morning. Descartes was a great philosopher as well as a mathematician and felt that mathematics should be a model for other branches of study. The following is a quote from his famous *Discours Sur la Methode:* "The long chain of simple and easy reasonings by means of which geometers are accustomed to reach conclusions of their most difficult demonstrations led me to imagine that all things, to the knowledge of which man is competent, are mutually connected in the same way, and that there is nothing so far removed from us as to be beyond our reach, or so hidden that we cannot discover it, provided only we abstain from accepting the false for the true, and always preserve in our thoughts the order necessary for the deduction of one truth from another."

René Descartes (1596–1650)

1.1 Introduction

The branch of mathematics in which different geometrical problems are solved with the help of algebraic method is called co-ordinate geometry. Famous French philosopher and mathematician Rene Descartes (1596–1650) is called the Father of co-ordinate geometry. According to Descartes Latin name Renatus Cartesius, co-ordinate geometry is also called cartesian geometry. Ancient Greek scientist made many researches about different kinds of curves especially on conic section. In the second century B.C. Appollonius made significant improvement in this subject.

In mathematics, the Cartesian Co-ordinate System (also called rectangular co-ordinate system) is used to determine each point uniquely in a plane through two numbers, usually called the x-co-ordinate or abscissa and the y-co-ordinate or ordinate of the point. To define the co-ordinates, two perpendicular directed lines (the x-axis and the y-axis) are specified, as well as the unit length, which is marked off on the two axes.

Once a co-ordinate system is defined over a plane, every point of that plane is uniquely represented by an ordered pair of real numbers. In 1637 Descartes presented the fundamental concept of co-ordinate geometry in La géométrie which is regarded as one of the most influential scientific texts of 17th century. The very first section of La géométrie is entitled "How the calculations of arithmetic are related to the operations of geometry". The second section describes, "How multiplication, division and extraction of square roots are performed geometrically". In the same time this system was developed independently by Pierre de Fermat, although Fermat did not publish his discovery which was published in 1679 after his death. An interesting observation is that neither Descartes nor Fermat considered the negative values for denoting the co-ordinates of a point in a plane and hence in the sense of modern time their works contain no Rectangular Cartesian Co-ordinate Systems.

These concepts are due to Newton and Leibnitz. Indeed, Newton is considered as the originator of the polar co-ordinate system. These ideas eventually gave birth to the subject of analytic geometry in two dimensions. In this branch a curve can be represented meaningfully by an equation $f(x, y) = 0$ in the sense that every point on the curve satisfies this equation. On the other hand if the ordered pair (x, y) satisfies the equation $f(x, y) = 0$, then the ordered

Quadrants	I XOY	II $X'OY$	III $X'OY'$	IV $Y'OX$
Sign of x	$+$	$-$	$-$	$+$
Sign of y	$+$	$+$	$-$	$-$
Sign of (x, y)	$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$

Note 1 : The ordinate of every point on x-axis is 0.

Note 2 : The abscissa of every point on y-axis is 0.

Note 3 : The abscissa and ordinate of the origin O (0,0) are both zero.

Note 4 : The co-ordinates of any point on x-axis are of the form $(x, 0)$.

Note 5 : The co-ordinates of any point on y-axis are of the form $(0, y)$.

1.6 Polar Co-Ordinates

1.6.1 Directed Angle

Two rays emerging from a certain fixed point O are said to be an angle, the point is called the vertex and two rays are called its sides. If one of the sides of an angle be taken as the initial side and the other as the terminal side then the angle is said to be directed angle. The magnitude of the least turning by which one side coincides with the other is the magnitude of the angle.

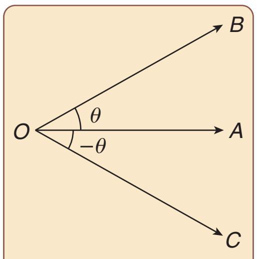

Fig ○ 1.4

The direction of directed angle is positive or negative according as the sense of the least turning by which the initial sides coincides with the terminal sides is anticlockwise or clockwise.

In Fig. 1.4, the angle formed by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, the measure of $\angle AOB = \theta$, but the measure of $\angle AOC = -\theta$.

1.6.2 Polar Co-ordinates of a Point

In polar co-ordinate system, the position of a point is determined with reference to a fixed point and a fixed half-line (i.e., a ray) in the plane. The fixed point is called as Pole and the fixed directed half-line is called initial line or polar axis.

Consider the point O as pole or origin and $\overrightarrow{OX}$ as the initial line or polar axis starting from O and extending indefinitely towards X (See Fig. 1.5).

and $\dfrac{\overline{PM}}{\overline{OP}} = \sin\theta$ or, $\overline{PM} = \overline{OP}\sin\theta$

or, $y = r\sin\theta$. (ii)

Again, $\overline{OP}^2 = \overline{OM}^2 + \overline{PM}^2$

or, $r^2 = x^2 + y^2$ or, $r = +\sqrt{x^2 + y^2}$ (iii)

and $\tan\theta = \dfrac{y}{x}$ or, $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$. (iv)

So, if the polar co-ordinates of a point P are known, then using (i) and (ii) we obtain the Cartesian co-ordinates (x, y) of point P. On the other hand, if the Cartesian co-ordinates (x, y) of P are given, then from (iii) we can determine the radius vector r of P, while its vectorial angle is determined from equation (iv) and the fact that $-\pi < \theta \leq \pi$.

1.8 Distance Between Two Points

Theorem 1 : The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Proof : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points in the plane. Let us assume that the points P and Q are both in 1st quadrant (for the sake of exactness).

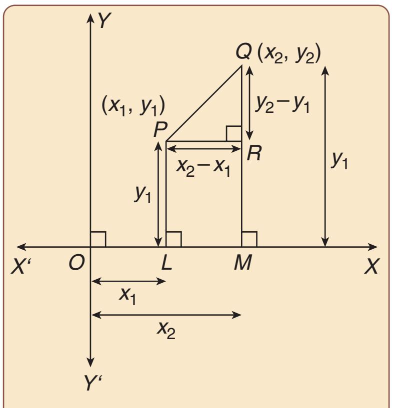

Fig ○ 1.8

From P and Q draw PL and QM perpendiculars to x-axis. From P draw PR perpendicular to QM and join PQ. Then

$$OL = x_1, OM = x_2, PL = y_1, QM = y_2.$$

$\therefore$ $PR = LM = OM - OL = x_2 - x_1$ and $QR = QM - RM = QM - PL = y_2 - y_1$.

Since PRQ is a right-angled triangle, therefore, by Pythagorean theorem.

$$(PQ)^2 = (PR)^2 + (QR)^2$$

$\therefore$ $|PQ| = \sqrt{(PR)^2 + (QR)^2}$ ($\because$ PQ is always positive)

$$= \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}.$$

Thus the distance $\boldsymbol{PQ}$ between the points $\boldsymbol{P(x_1, y_1)}$ and $\boldsymbol{Q(x_2, y_2)}$ is given by $\boldsymbol{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$.