where ${}_n^Z B_x^f$ = number of female born to women in the age group (x to $x+n$) in the calender year Z and

${}_n W_x$ = number of women in the age group (x to $x+n$).

Further, $\frac{{}_n L_x}{l_0}$ will represent the average number of years lived by the Cohort of newly born potential mother in the age group (x to $x+n$) over which the probability of having a female baby, $\frac{{}_n^Z B_x^f}{{}_n W_x}$, per year will be applied for a period of $\frac{{}_n L_n}{n}$ for K mothers will give rise to (3.1.8).

The Physical Interpretation of the NRR:

The NRR per K women will imply the total number of female babies born by K number of newly born female infants during their fertility span subject to their survival at the present mortality schedule throughout the fertility span.

Thus NRR can be taken as a growth Index. If NRR $> K$, we can conclude that the population is increasing and if NRR $< K$ we may conclude that the population is decreasing. In other words, K number of potential mothers will fail to produce sufficient number of future mothers to maintain the same level.

Hence, for rough work NRR may be used as the measure of replacement index of the population; replacement index is the quantitative measurement of one generation replaced by other generation.

Note that the following assumptions have been made for the estimation of NRR:

(1) The same survival factor (i.e. mortality) as given in the current life table valid for the previous decade will be applicable for the entire newly born babies throughout their life time i.e.

$$\left(\frac{l_x}{l_0}\right) = \frac{l(x,t)}{l(0,t)} = \text{constant i.e. independent of } t \tag{3.1.9}$$

(2) Same fertility pattern as available in the latest fertilely schedule. will be applicable for the newly born babies throughout their life time.

(3) the sex ratio $\left(= \frac{\text{number of males}}{\text{total population}}\right)$ should remain the same over all the years to come.

But the above assumptions are not always justified. If the mortality rate has a decreasing trend then the actual NRR $\geq$ estimated NRR on the basis of the above assumption and if the mortality is increasing then a reverse conclusion holds good.

3.2 A Relationship Between Crude Birth Rate, General Fertility Rate and Total Fertility Rate

Let, $c(x;t)$ = Observed proportion of females in the age group (x to $x+1$) at time t, (3.2.1)

and $f(x;t)$ = Observed proportion of females giving birth to female children in the age group (x to $x+1$) at time t (3.2.2)

$$\text{Then, } \int_\alpha^\beta f(x;t)dx = \text{Estimated total fertility rate at time } t = \hat{T}_f(t), \text{ say} \tag{3.2.3}$$

$$\text{and} \quad \int_\alpha^\beta c(x;t)f(x;t)dx = \text{Estimated female birth rate at time } t$$
$$= \hat{B}_f(t), \text{ say,} \tag{3.2.4}$$

where α and β represent the lower and the upper bound of the child bearing age. The correlation between $c(x;t)$ and $f(x;t)$, at a given t, is given by

$$\text{Cor } (c(x;t), f(x;t)|t) = \frac{E(c(x;t)f(x:t)|t) - E(c(x;t)|t)E(f(x;t)|t)}{\sqrt{\text{Var}(c(x;t)|t)}\sqrt{\text{Var}(f(x;t)|t)}}$$
$$= r_{c,f|t}, \text{ say} \tag{3.2.5}$$

We have, $E(c(x;t)f(x;t)|t) = \int_\alpha^\beta [c(x;t)f(x;t)|t]\phi_x dx,$

where ϕ_x is the probability distribution of the r.v. X, the child bearing age.

Also, $E(c(x;t)|t) = \int_\alpha^\beta [c(x;t)|t]\phi_x dx,$

$$E(f(x;t)|t) = \int_\alpha^\beta [f(x;t)|t]\phi_x dx,$$

$$\text{Var}(c(x;t)|t) = \int_\alpha^\beta [c^2(x;t)|t]\phi_x dx - [E(c(x;t)|t)]^2 = \sigma^2_{c/t},$$

$$\text{Var}[f(x;t)|t] = \int_\alpha^\beta [f^2(x;t)|t]\phi_x dx - [E(f(x;t)|t)]^2 = \sigma^2_{f/t}.$$

Let us assume that X is uniformly distributed in (α, β) i.e.

$$\phi_x = \frac{1}{\beta - \alpha}; \ \alpha \le x \le \beta = 0 \text{ otherwise} \tag{3.2.6}$$

Given $\hat{r}_{c,f|t} = \text{Cor}(c(x;t), f(x;t)|t) = \left[\frac{1}{\beta-\alpha}\int_\alpha^\beta [c(x;t)f(x;t)|t]dx\right.$

$$\left. - \frac{1}{\beta-\alpha}\int_\alpha^\beta [c(x;t)|t]dx \frac{1}{\beta-\alpha}\int_\alpha^\beta [f(x;t)|t]dx\right] \Big/ \sigma_{c|t}\sigma_{f|t}$$

$$\text{where } \sigma_{c|t} = \sqrt{\left[\left\{\frac{1}{\beta-\alpha}\int_\alpha^\beta [c(x,t)|t\right]^2 dx - \frac{1}{\beta-\alpha}\left[\int_\alpha^\beta [c(x,t)|t]dx\right]^2}$$

$$\text{and} \quad \sigma_{f|t} = \sqrt{\left[\left\{\frac{1}{\beta-\alpha}\int_\alpha^\beta [f(x,t)|t\right]^2 - \frac{1}{\beta-\alpha}\left[\int_\alpha^\beta [f(x,t)|t]dx\right]^2}$$

$$\Rightarrow \quad r_{c,f|t\sigma_{c|t}\sigma_{f|t}} = \left\{\frac{1}{\beta-\alpha}\int_\alpha^\beta [c(x;t)f(x;t)|t]dx\right.$$
$$- \frac{1}{(\beta-\alpha)^2}\int_\alpha^\beta [c(x;t)|t]dx \int_\alpha^\beta [f(x,t)|t]dx$$
$$= \frac{1}{\beta-\alpha}\hat{B}_f(t) - \frac{\hat{T}_t(t)}{(\beta-\alpha)^2}$$

where $\hat{B}_f(t) = \int_\alpha^\beta [c(x,t)f(x,t)|t]dx$

and $\hat{T}_f(t) = \int_\alpha^\beta [c(x,t)|t] \int_\alpha^\beta [f(x,t)|t]dx$

Again, estimated General fertility rate (GFR) is given by

$$\hat{G}_F(t) = \frac{N\int_\alpha^\beta [f(x;t)|t]dx}{N\int_\alpha^\beta [c(x;t)|t]dx} \text{ from (3.23) and (3.24)} \tag{3.2.7}$$

$$\Rightarrow \int_\alpha^\beta [c(x;t)|t]dx = \frac{\hat{T}_f(t)}{\hat{G}_f(t)} \tag{3.2.8}$$

Putting (3.5) in (3.4)

$$\Rightarrow [\hat{r}_{c,f|t}]\hat{\sigma}_{c|t}\hat{\sigma}_{f|t} = \frac{\hat{B}_f(t)}{\beta-\alpha} - \frac{\hat{T}_f(t)}{(\beta-\alpha)^2}\frac{\hat{T}_f(t)}{\hat{G}_F(t)} \tag{3.2.9}$$

$$\Rightarrow (\beta-\alpha)\hat{r}_{c,f|t}\hat{\sigma}_{c|t}\hat{\sigma}_{f|t} = \hat{B}_f(t) - \frac{(\hat{T}_f(t))^2}{\beta-\alpha}\frac{1}{\hat{G}_F(t)} \tag{3.2.10}$$

$$\Rightarrow \hat{B}_f(t) = \left[\hat{r}_{c,f|t}\hat{\sigma}_{c|t}\hat{\sigma}_{f|t}(\beta-\alpha) + \frac{(\hat{T}_f(t))^2}{(\alpha-\beta)}\frac{1}{\hat{G}_F(t)}\right] \tag{3.2.11}$$

which established a relation between birth rate, total fertility rate and general fertility rate.

A special case

$$\text{if} \quad \hat{r}_{c,f|t} = 0 \Rightarrow \hat{B}_f(t) = \frac{(\hat{T}_f(t))^2}{\beta-\alpha}\frac{1}{\hat{G}_F(t)} \tag{3.2.12}$$

3.3 Mathematical Models on Fertility

The origin of the systematic development of Stochastic models in India, representing the fertility behaviour under uncontrolled condition of fertility or a description of the Human Reproductive Process, perhaps has its foundation is the work of Dandekar (1955). Dandekar while formulating an appropriate probability model of the number of births in a given marital exposure of duration t developed special interruped probability distributions known as 'Modified Binomial' 'Modified and Poisson' distributions.

3.3.1 Dandekar's Modified Binomial Distribution

Dandekar's Modified Binomial distribution is based on the following assumptions:

(i) Probability of a conception (Vis-a-vis a birth, assuming a one-to-one correspondence between a conception and a birth) is p in every trial (every trial is assumed to be of duration of one month approximately which is the interval between two consecutive ovulatory cycles).

(ii) Given that there is a success (a conception leading to a birth), the probability of a further success in another π number of trials (π is an integer) inclusive of the trial in which a success took place is zero i.e. in π number of trials then can be almost one success.

4

Techniques of Demographic Analysis: Population Growth Indices

4.0 Introduction

Given the indices of mortality and fertility as discussed in Chapter-2 and Chapter-3 for a natural population, a question that naturally arises is whether the tendency of the given population is to increase, decrease or to remain more or less stationary over time. Therefore, before any idea about the growth of a population is taken it is necessary to evolve proper indices of population growth. We have seen in Chapter-3 that a fairly good index of population growth is given by "Net Reproduction Rate". But the validity of the Net Reproduction Rate is again subjected to the limitation of fertility and mortality conditions maintaining 'status quo'. Besides that the Net Reproduction Rate essentially reflects the growth in female population. The parallel index for male is difficult to construct. Thus to estimate growth of a population, a good approach would be to evolve certain theoretical models (deterministic or stochastic) describing the growth of populations. Growth parameter may often be estimable from the model itself.

4.1 Measurements of Population Growth

Let us take the simplest situation in which we have a population of N individuals and we assume that population remains closed for Migration i.e. there is neither immigration nor emigration.

The modeling for natural populations have three basic characterizations as follows:

(i) The population over time may show that the average density of population is being maintained at a constant level over a long period of time, unless there is a major environmental change. In other words, population under such a setup neither die out nor explode. This, phenomenon, is called 'Balancing', by Nicholson (1957).

(ii) In the second characterization, the growth of a population need not necessarily remain at a constant level but the same may fluctuate around a constant mean value randomly.

(iii) Finally, a third characterization is given by superimposition of a random cycle of oscillation on the type of random variation, already considered in the second type of characterization, Thus the third characterization is a generalization of the second.

With this setup we propose to develop a simple deterministic model. Here, we ignore such factors as environmental conditions etc. while developing a deterministic model. Further, in a deterministic model the probabilistic considerations relating to the variation of N (which is an integer) is ignored and one can reasonably assume N to be a continuous variable (vide Moran (1961)).

Let $N(t)$ be the population at time t, $B(t)$ and $D(t)$ be the birth rate and death rate, respectively, at time t. Then the relative growth of the population at time t is

$$\frac{1}{N(t)}\frac{dN(t)}{dt} = B(t) - D(t) \tag{4.1.1}$$

$$= r(t), \quad \text{(say)} \tag{4.1.2}$$

Note that $B(t)$ and $D(t)$ are defined w.r.t. the whole population, since we are ignoring the distinction between male and female population.

If $r(t) = r$, a constant independent of t, then from (4.1.2)

$$\frac{1}{N(t)}\frac{dN(t)}{dt} = r, \tag{4.1.3}$$

$$\Rightarrow \quad \frac{d\log N(t)}{dt} = r \quad \Rightarrow \quad \log N(t) = r\cdot t + a \text{ (an integration constant)}$$

$$\Rightarrow \quad N(t) = Ae^{rt}, \quad \text{where} \quad A = e^{a} \tag{4.1.4}$$

This model (4.1.4) was given by Malthus (1798). This is called Malthusian geometric law of population growth.

Here as $t \to \infty$, $N(t) \to \infty$, which is unrealistic, due to the resources and geographic conditions of the region.

4.2 A Population Density Dependent Growth Model

We may note that the assumptions in (4.1.1) that $B(t)$ and $D(t)$ are independent of N, are not realistic i.e. the reason of a population being at a given level must be explained by the density of the population. In view of this we write (4.1.1) as

$$\left.\begin{aligned}\frac{1}{N(t)}\frac{dN(t)}{dt} &= B(N,t) - D(N,t)\\ &= r(N,t)\end{aligned}\right\} \tag{4.2.1}$$

4.2.1 The Logistic Model for Population Growth

Verhulst (1838) took into account the limitation of Malthus model (4.1.4) and postulated that the relative growth rate $r(t)$ is a decreasing function of $N(t)$.

If M is the maximum population that a region can support, then according to Verhulst $r(t)$ may be taken as

$$r(N,t) = r\left[1 - \frac{N(t)}{M}\right], \tag{4.2.2}$$

where $r(>0)$ and M are constants.

So, (4.2.1) becomes

$$\frac{1}{N}\frac{dN}{dt} = r[1 - a\cdot N], \quad \text{where} \quad N = N(t) \quad \text{and} \quad a = 1/M. \tag{4.2.3}$$

$$\Rightarrow \quad \frac{dN}{N(1-a\cdot N)} = r\ dt \quad \Rightarrow \quad \left[\frac{1}{N} + \frac{a}{1-a\cdot N}\right] dN = r\cdot dt$$

Integrating both side we get

$$\log N - \log(1 - a \cdot N) = r \cdot t + c \text{ (an integration constant).}$$

$$\Rightarrow \quad \log \frac{N}{1 - a \cdot N} = r \cdot t + c \Rightarrow \frac{N}{1 - a \cdot N} = A \cdot e^{r \cdot t}, \quad \text{where} \quad A = e^c$$

$$\Rightarrow \quad N = \frac{1/a}{1 + \frac{1}{a \cdot A} e^{-r \cdot t}}$$

$$\text{or} \quad N(t) = \frac{k}{1 + b.e^{-r \cdot t}} \quad \text{where} \quad k = 1/a \quad \text{and} \quad b = 1/(a \cdot A). \tag{4.2.4}$$

This is a solution (4.2.3).

The equation (4.2.4) is known as Logistic law of Growth; k, b and r are three parameters of the distribution. This is discussed in the following sections.

The initial value at $t = 0$, i.e. $N(0)$ is given by

$$N(0) = \frac{k}{1 + b} \tag{4.2.5}$$

The limiting value of $N(t)$ as $t \to \infty$, converges as

$$\lim_{t \to \infty} N(t) = N(\infty) = k, \text{ from (4.2.4).} \tag{4.2.6}$$

Now, from (4.2.5) and (4.2.6)

$$N(0) < N(\infty) \ \Rightarrow \ 1 + b > 1 \Rightarrow b > 0$$

4.2.2 Properties of Logistic Curve

Let us denote $N(t) = y(t) = y$, then the equation of logistic curve (4.2.4) becomes

$$y(t) = \frac{k}{1 + b.e^{-r \cdot t}} \tag{4.2.7}$$

$$\text{Then} \quad \frac{dy}{dt} = \frac{k \cdot b \cdot r \cdot e^{-r \cdot t}}{(1 + b \cdot e^{-r \cdot t})^2} = \frac{b.e^{-r \cdot t} r \cdot k}{(1 + b \cdot e^{-r \cdot t})(1 + b \cdot e^{-r \cdot t})}.$$

Now, from (4.2.7)

$$1 + b.e^{-r \cdot t} = \frac{k}{y} \Rightarrow b.e^{-r \cdot t} = \frac{k}{y} - 1$$

$$\Rightarrow \quad \frac{b.e^{-r \cdot t}}{1 + b.e^{-r \cdot t}} = \left(\frac{k}{y} - 1\right) \Big/ \left(\frac{k}{y}\right)$$

$$\therefore \quad \frac{dy}{dt} = r \cdot y \left(\frac{k}{y} - 1\right) \Big/ \left(\frac{k}{y}\right) = r \cdot y \left(1 - \frac{y}{k}\right) \tag{4.2.8}$$

$$\text{Hence,} \quad \frac{d^2 y}{dt^2} = r \cdot \frac{dy}{dt} \left(1 - 2\frac{y}{k}\right) = r^2 . y \left(1 - \frac{y}{k}\right) \left(1 - 2\frac{y}{k}\right) \tag{4.2.9}$$

From(4.2.8), $\frac{dy}{dt} = 0$ at $y = 0$ and $y = k$ i.e. at $t = -\infty$ and $t = \infty$, from (4.2.7).

Thus $y = 0$ and $y = k$ are two asymptotes of the logistic curve (4.2.7).

Further, from (4.2.8)

$$0 < y < k \quad \Rightarrow \quad \frac{dy}{dt} > 0$$

i.e. logistic curve $y(t)$ is a monotonically increasing curve.

Again, from (4.2.9),

$$\frac{d^2y}{dt^2} = 0 \quad \Rightarrow \quad 2\frac{y}{k} = 1 \quad \Rightarrow \quad y = k/2, \text{is the point of inflection.}$$

The logistic curve $y(t)$ given by (4.2.7) may be represented as given below; for $y < k$, it is a S-shaped curve.

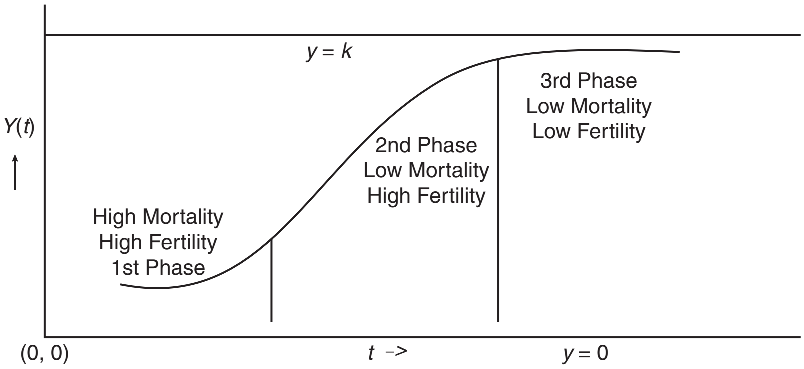

Fig. 4.1

4.2.3 A Critical Review of the Logistic Model

Verhulst (1838) evolved the Logistic curve while experimenting with growth of the insects under controlled environmental condition. Also Logistic curve derived by other similar assumption has, in the past, given good fit to the population data of several countries as well as to other kinds of data. As a matter of fact, the Logistic model with three parameters often gives a closed fit to the population data taken over periods of time. But as cautioned by Fellar (1940) closed fit of the data with the logistic does not necessarily give any evidence that the underlying process of growth follows a logistic pattern. As such the population projection by fitting of logistic curve remains often a very questionable issue. A result due to Leslie (1948) in this respect may appear to be quite interesting. Leslie has shown that if the age specific fertility and mortality rates are also density dependent i.e. they depend on the size of the population then the population growth rate will acquire a logistic pattern provided the initial age distribution from which we start is stable or stationary. The implication of this result is that if the initial age distribution is not stable then the growth rate will not acquire a logistic from. But Indian population, which is far from having a stable age distribution, has given quite good fit to the logistic distribution. This is quite misleading Demographically. It is, therefore, advisable to examine critically the theoretical premises in support of a model relating to a population growth curve, (especially logistic) before the same is employed for the measurement of the population growth or projection purpose.

Further, it is true that a logistic model for predicting the growth of a population has all the defects and limitations, which a single equation model has while predicting the course of a population.

4.3 Methods of Fitting Logistic Curves

In order to fit the Logistic curve (4.2.7) we are to estimate the parameters k, b and r. There are many methods of fitting a Logistic curve, we discuss here a few more common ones.

Now from (2.7.7),

$$T_0 = \int_0^\infty l(x)dx \tag{4.4.50}$$

$\therefore \quad d = l_0/T_0 = 1/e_0^0$, from (2.7.10)

where e_0^0 is the complete expectation of life. Thus the life table death rate is usually denoted by $1/e_0^0$.

(iii) Determination of Q_0 under stable population analysis.

Consider a general formulation

$$B(t) = G(t) + \int_0^t B(t_x)\psi(x)dx, \tag{4.4.51}$$

where $G(t)$ is the number of births to women already born at time $t = 0$.

The term $G(t) = 0$ for $t \geq \beta$ i.e. no birth can occur to women of age $\geq \beta$.

or $$G(t) = B(t) - \int_0^t B(t-x)p(x)\cdot i(x)dx. \tag{4.4.52}$$

Taking Laplace Transform of (4.4.52) on both sides

$$\Rightarrow \quad L(G(t)) = \int_0^\infty \exp(-r_0 t)G(t)dt = \int_0^\beta \exp(-r_0 t)G(t)dt$$

$$(\because G(t) = 0 \,\forall\, t \geq \beta)$$

$$= \int_0^\beta \exp(-r_0 t)\left[B(t) - \int_0^t B(t-x)\psi(x)dx\right], \text{ using (4.4.51)}$$

$$= \int_0^\beta \exp(-r_0 t)\left[Q_0 \exp(r_0 t) + \sum_{n=1}^\infty Q_n \exp(r_n t)\right.$$

$$\left. - \int_0^t \left\{Q_0 \exp[r_0(t-x)] + \sum_{n=1}^\infty Q_n \exp[r_n(t-x)]\right\}\psi(x)dx\right] dt$$

using (4.4.18)

$$= \int_0^\beta \exp(-r_0 t)\left[Q_0 \exp(r_0 t) + \sum_{n=1}^\infty Q_n \exp(r_n t)\right.$$

$$\left. - \int_0^t Q_n \exp[r_0(t-x)]\psi(x)dx + \int_0^t \sum_{n=1}^\infty Q_n \exp[r_n(t-x)]\psi(x)dx\right] dt$$

$$= \int_0^\beta \exp(-r_0 t)\left[Q_0 \exp(r_0 t) - \int_0^t Q_0 \exp\{r_0(t-x)\}\psi(x)dx\right] dt + C_0' \tag{4.4.53}$$

where $$C_0' = \int_0^\beta \exp(-r_0 t)\left[\sum_{n=1}^\infty Q_n \exp(r_n t)\right.$$

$$\left. - \int_0^t \left\{\sum_{n=1}^\infty Q_n \exp[r_n(t-x)]\psi(x)dx\right\}\right] \tag{4.4.54}$$

Therefore, $L(G(t)) = \int_0^\beta \left\{ Q_0 - \int_0^t Q_0 \exp(-r_0 x)\psi(x)dx \right\} dt + C_0'$

$$= Q_0 \int_0^\beta \left\{ 1 - \int_0^t \exp(-r_0 x)\psi(x)dx \right\} dt + C_0' \qquad (4.4.55)$$

Since r_0 is the real root of the equation

$$\int_\alpha^\beta e^{-rx} p(x) f(x) dx = \int_0^\infty e^{-rx} p(x) f(x) dx = 1$$

or

$$\int_0^\beta e^{-rx}\psi(x)dx = 1,$$

we may write

$$L(G(t)) = Q_0 \int_0^\beta \left\{ \int_0^\beta \exp(-r_0 x)\psi(x)dx - \int_0^t \exp(-r_0 x)\psi(x)dx \right\} dt + C_0'$$

$$= Q_0 \int_0^\beta \int_t^\beta \exp(-r_0 x)\psi(x)dx\ dt + C_0' \qquad (4.4.56)$$

Next we shall change the order of integration in the above double integral.

Note that in the case given above sums of the type of vertical strips given in the Fig. 4.2 were taken. Now we shall take sums of the horizontal type of strips. The relevant limits are thus

$$0 \le x \le \beta \text{ and } 0 \le t \le x$$

Hence (4.4.56) becomes

$$L(G(t)) = Q_0 \int^\beta \int_0^x \exp(-r_0 x)\psi(x)dt\ dx + C_0'$$

$$= Q_0 \int_0^\beta \left\{ \exp(-r_0 x)\psi(x) \int_0^x dt \right\} dx + C_0' \qquad (4.4.57)$$

$$= Q_0 \int_0^\beta \exp(-r_0 x\psi(x) \cdot x dx + C_0' \qquad (4.4.58)$$

$$\Rightarrow \quad Q_0 = \frac{L(G(t))}{\int_0^\beta x \exp(-r_0 x)\psi(x)dx}$$

$$= \frac{\int_0^\beta \exp(-r_0 t) G(t) dt}{\int_0^\beta x \exp(-r_0 x)\psi(x)dx}, \qquad (4.4.59)$$

provided $C_0' = 0$.

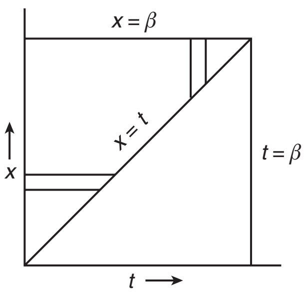

Fig. 4.2

Now to show that $C_0' = 0$, we see that C_0' is made up of sums of such terms as:

$$C_{0n}' = Q_n \int_0^\beta \exp(-r_0 t) \left[\exp r_n t - \int_0^t \exp(r_n(t-x)]\psi(x)dx \right] dt,\ n \neq 0$$

$$= Q_n \int_0^\beta \left[\exp[r_n - r_0)t] - \int_0^t \exp[(r_n - r_0)t] \exp(-r_n x)\psi(x)dx\right] dt$$

$$= Q_n \int_0^\beta \exp[(r_n - r_0)t] \left\{1 - \int_0^t \exp(-r_n x)\psi(x)dx\right\} dt.$$

We put $\int_0^\beta \exp(-r_n x)\psi(x)dx = 1.$

Again since r_n is also a root of the Lotka's integral equation, we get

$$C'_{0n} = Q_n \int_0^\beta \exp[(r_n - r_0)t] \left\{\int_0^\beta \exp(-r_n x)\psi(x)dx - \int_0^t \exp(-r_n x)\psi(x)dx\right\} dt$$

$$= Q_n \int_0^\beta \exp[(r_n - r_0)t] \left\{\int_t^\beta \exp(-r_n x)\psi(x)dx\right\} dt \tag{4.4.60}$$

Again changing the order of integration

$$C'_{0n} = Q_n \int_0^\beta \left\{\exp(-r_n x)\psi(x) \int_0^x \exp[(r_n - r_0)t]dt\right\} dx$$

$$= \frac{Q_n}{r_n - r_0} \int_0^\beta \exp(-r_n x)\psi(x)[\exp\{r_n - r_0)\}x - 1]dx$$

$$= \frac{Q_n}{r_n - r_0} \left\{\int_0^\beta \exp(-r_0 x)\psi(x)dx - \int_0^\beta \exp(-r_n x)\psi(x)dx\right\}$$

$$= 0, \tag{4.4.61}$$

because r_u and r_0 are both roots of Lotka's integral equation (4.4.20)

i.e. $$\int_0^\beta \exp(-r_0 x)\psi(x)dx = \int_0^\beta \exp(-r_n x)\psi(x)dx = 1$$

Again $C'_0 = \sum_{n=1}^{\infty} C'_{0n}$, and each $C'_{0n} = 0$, from (4.4.61)

Therefore, $$C'_0 = 0. \tag{4.4.62}$$

We have, $$Q_0 = \frac{\int_0^\beta \exp(-r_0 t)G(t)dt}{\int_0^\beta x\exp(-r_0 x)\psi(x)dx}, \text{ from (4.4.59)} \tag{4.4.63}$$

If we know $G(t)$ and the rate of survivorship f and fertility $p(x)[\psi(x) = p(x) \cdot f(x)]$ then one can calculate Q_0 from r_0.

A special case

Let us take the daughters of B_0 births occurred at $t = 0$.

Then $G(t) = B_0(t)p(t)f(t) = B_0\psi(t)$